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WAVE PROPAGATION IN ELASTIC BODIES WITH RESTRICTED SHEAR*

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Abstract—This paper examines the propagation of waves in unbounded elastic bodies within which shear is restricted in a specific, preferred plane. It is shown that three waves propagate normal to this plane but only two propagate in any other direction. The singularities arising on the associated slowness surface are elucidated by considering materials for which the modulus of shear is high in a preferred plane. We show that the innermost sheet of the slowness surface is then a thin pencil shape normal to the preferred plane. This paper concludes with an investigation of energy propagation.

1. INTRODUCTION

Internal constraints have been used for many years as a mathematical idealization in the modelling of various classes of elastic material. When an elastic body is constrained a restriction is placed upon the class of deformations the body may undergo. In the modelling of solid materials the two most commonly encountered constraints are those of inextensibility and incompressibility. Associated with any internal constraint is a workless reaction stress. This stress is not determined by the deformation but must be chosen so that the equations of motion and boundary conditions can be satisfied. The number of non-zero eigenvalues of the reaction stress tensor is called the dimension of the constraint, see Pipkin (1976). A spectral representation of this tensor may be used to offer a physical interpretation of this dimensional classification. A one-dimensional constraint, such as inextensibility, may be defined as one for which the reaction stress acts only in a single fixed direction. For inextensibility this stress is a tension acting in the fibre direction. A three-dimensional constraint is characterized by having an associated reaction stress acting throughout the whole space. The reaction stress associated with the three-dimensional constraint of incompressibility is a hydrostatic pressure.

In comparison with the two classes of constraint already mentioned, two-dimensional constraints have received little attention in the literature. The two-dimensional constraint of restricted shear has attracted some attention because of its possible applications to laminated materials, see Spencer (1972). A body under such a constraint may only be deformed in such a way that the angle between two specific directions does not change. The associated reaction stress is a shear stress acting in the plane of these two directions. Furthermore, any two-dimensional constraint is characterized by having a reaction stress acting in a specific plane. A study of surface wave propagation in elastic bodies subject to restricted shear has been carried out by Whitworth and Chadwick (1984) and the propagation of acceleration waves in such materials was mentioned briefly by Chen and Gurtin (1974). Recently, this constraint has also received some attention because of its possible applications to twinned crystals, see Ericksen (1986) and Scott (1990). The constraint is an idealization used to model a class of materials for which a preferred plane exists. Within this plane a large shear stress is required to produce a small shear strain. The implication is that the modulus of shear within this plane is high.

This paper begins with a review of the basic theory of wave propagation in elastic bodies subject to a single arbitrary constraint. This broad theoretical framework is particularized in Section 3(i) to investigate the propagation of waves in elastic bodies with restricted shear. It is shown that the number of waves able to propagate in such materials is a discontinuous function of the direction of propagation. This phenomenon arises because three waves

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propagate normal to the preferred plane but only two propagate in any other direction. In order to elucidate the singularities occurring on the associated slowness surface and provide a more physically realistic model, the constraint is relaxed in Section 3(ii). In this section we consider materials in which a preferred plane exists within which the modulus of shear is an order of magnitude higher than any other material constant. This large modulus of shear is then used to obtain an approximate solution of the propagation condition in the form of a power series. It is then possible to show that the innermost sheet of the slowness surface is a thin pencil shape normal to, and through, the preferred plane. The limit to the exact constraint is obtained by letting the modulus of shear in the preferred plane tend to infinity. When this limit is taken we show that the pencil shape tends to a straight line along the normal to the plane. This then explains how it is possible for three waves to propagate in this direction when the body is subject to the exact constraint.

In the final section the propagation of energy is considered. We begin by verifying that the incremental energy components, see Chadwick *et al.* (1985), satisfy various energy relations. The energetics of the wave associated with the innermost sheet of the slowness surface are examined in some detail. The power series solutions of the propagation condition are used to show that the magnitude of the kinetic energy associated with this wave depends upon both the modulus of shear in the preferred plane and the wave amplitude. Finally, it is shown that to leading order the energy velocity vector lies in the preferred plane.

Throughout this paper we employ both direct and indicical notations, vector and tensor components being referred to an arbitrary orthogonal basis. All subscripts take the values 1, 2, 3, the summation convention applies to repeated suffixes and a superimposed dot implies differentiation with respect to time.

2. BASIC EQUATIONS FOR AN ARBITRARY CONSTRAINT

Any arbitrary internal elastic constraint may be expressed in the form

$$\lambda(\mathbf{F}) = 0, \tag{1}$$

in which the tensor $\mathbf{F}$ is the deformation gradient, see Chadwick (1976, p. 145). It is convenient for problems involving internal constraints to introduce the pseudo strain energy function

$$W(\mathbf{F}) = W_0(\mathbf{F}) + \alpha\lambda(\mathbf{F}), \tag{2}$$

where $W_0(\mathbf{F})$ generates the constitutive part of the stress totally determined by the deformation, $\lambda(\mathbf{F})$ the workless reaction stress associated with the constraint and α is a scalar multiplier, see Scott (1975). The total Cauchy stress may then be obtained, without need to consider the constaint terms separately, by employing the relation

$$\sigma_{ij} = J^{-1}F_{ip}\frac{\partial W}{\partial F_{jp}}, \tag{3}$$

in which $J = \det \mathbf{F}$. Similarly, the fourth-order elasticity tensor $\mathbf{B}$ involves terms arising from the constraint and may be expressed in the component form

$$B_{ijkl} = J^{-1}F_{ip}F_{kq}\frac{\partial^2 W}{\partial F_{jp}\,\partial F_{lq}}. \tag{4}$$

We now consider a homogeneous elastic body B with a natural undistorted state B_u and reference state B_r; the deformation $B_u \rightarrow B_r$ being homogeneous. The equations of small-amplitude motions superimposed upon this large static pre-strain are given by

$$\rho \ddot{U}_j = B_{ijkl} U_{l,ik} + \beta_{,i} N_{ij}, \tag{5}$$

in which ρ is the material density, U the displacement, β a small time-dependent perturbation to the scalar multiplier α, N the reaction stress and both $\boldsymbol{B}$ and N are evaluated in B_r, see Chadwick *et al.* (1985).

It will be assumed that U and β may be expressed in the forms

$$U_i = \varphi(\mathbf{x}\cdot\mathbf{n} - vt)e_i, \quad \beta = \varphi'(\mathbf{x}\cdot\mathbf{n} - vt)q, \tag{6}$$

the prime implying differentiation with respect to argument. These expressions characterize a plane wave propagating in a direction defined by the unit vector $\mathbf{n}$, with speed v and unit polarization vector e. If these forms U and β are inserted into the propagation condition (5) the following linear equations are obtained:

$$Q_{jl} e_l + q \nu_j = \rho v^2 e_j, \tag{7}$$

where the symmetric acoustical tensor Q and constraint vector $\boldsymbol{\nu}$ are defined through the component relations

$$Q_{jl} = B_{ijkl} n_i n_k, \quad \nu_j = N_{ij} n_i. \tag{8}$$

In addition, the polarization vector e satisfies the relation

$$\boldsymbol{\nu}\cdot\mathbf{e} = 0,$$

this relation is a linearized form of the constraint function (1), see Chadwick *et al.* (1985, Sections 3 and 4). It is clear that for non-vanishing, $\boldsymbol{\nu}$, e is normal to the constraint vector. In the case $\mathbf{e}\cdot\boldsymbol{\nu} = 0$, $\boldsymbol{\nu} \neq \mathbf{0}$ and eqn (7) implies that two plane waves are able to propagate. However, when $\boldsymbol{\nu} = \mathbf{0}$ this equation is very similar to the classical unconstrained propagation condition and it is therefore possible for three plane waves to propagate, see Truesdell and Noll (1985, pp. 71, 73). The only possible difference between (7) and the unconstrained case when $\boldsymbol{\nu} = \mathbf{0}$ is possible terms in Q arising from the constraint.

3. RESTRICTED SHEAR

(i) *The constraint*

A body subject to the constraint of restricted shear may only be deformed in such a way that the angle between two specified directions remains constant. Suppose that in the undistorted state B_u the directions of these two line elements are denoted by the unit vectors A and B. The same line elements in B_r, denoted by $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$, are related to those in B_u through the relations

$$\bar{a}_i = F_{ip} A_p, \quad \bar{b}_i = F_{ip} B_p. \tag{9}$$

If unit vectors parallel to $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are denoted by a and b, the constraint function associated with this constraint may be expressed as

$$\lambda(\mathbf{F}) = a_i b_i - A_i B_i = 0. \tag{10}$$

Using eqn (3), the reaction stress associated with this constraint may be shown to be any scalar multiple of

$$N_{ij} = a_i b_j - b_i a_j - \tfrac{1}{2}(\mathbf{a}\cdot\mathbf{b})(a_i a_j + b_i b_j). \tag{11}$$

For simplicity, it will be assumed that $\mathbf{a}\cdot\mathbf{b} = 0$, in which case that part of the elasticity tensor arising from the constraint $\boldsymbol{B}^c$ takes the form

$$B^{c}_{ijkl} = (a_i b_k + a_k b_i)\delta_{lj} - (a_i b_j + b_i a_j)(b_k b_l + a_k a_l) \\ - a_i a_j a_k b_l - a_i a_j b_k a_l - b_i b_j a_k b_l - b_i b_j b_k a_l. \quad (12)$$

It is possible to use the reaction stress (11) and the elasticity tensor (12) in conjunction with eqns (8) to deduce that

$$Q^{c}_{jl} = 2(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n})c_j c_l - ((\mathbf{a}\cdot\mathbf{n})^2 + (\mathbf{b}\cdot\mathbf{n})^2)(a_j b_l + b_j a_l). \quad (13)$$

$$v_i = (\mathbf{b}\cdot\mathbf{n})a_i + (\mathbf{a}\cdot\mathbf{n})b_i, \quad (14)$$

where $\mathbf{Q}^c$ is that part of $\mathbf{Q}$ associated with the constraint, $\boldsymbol{v}$ the constraint vector and $\mathbf{c} = \mathbf{a} \wedge \mathbf{b}$. From eqn (14) it is readily deduced that the constraint vector vanishes when $\mathbf{n}$ is normal to the plane containing $\mathbf{a}$ and $\mathbf{b}$, i.e. when $\mathbf{n} = \mathbf{c}$. Equations (7) and (13) may now be invoked to observe that in this case the propagation condition is *exactly* that of the unconstrained problem. This imples that provided the constitutive part of $\boldsymbol{B}$ satisfies some criterion for physically reasonable response, such a strong ellipticity, three homogeneous plane waves propagate in this direction. This contrasts with the two, not necessarily homogeneous waves, which are able to propagate in any other direction.

(ii) *The near-constraint*

In the previous section it was shown that the number of waves able to propagate in an elastic body with restricted shear is a discontinuous function of the direction of wave propagation. This phenomenon arises because three waves propagate normal to the preferred plane but only two can propagate in any other direction. A similar situation arises in the case of elastic bodies reinforced by inextensible fibres, see Chen and Gurtin (1974). For this particular problem it is possible for three plane waves to propagate in any direction within the plane normal to the fibre direction. The directions within which three plane waves propagate are usually referred to as exceptional, see Scott (1975). The existence of exceptional directions for a given constraint implies that singularities exist on the slowness surface of an elastic body subject to that constraint. The class of constrained elastic media for which singularities exist is restricted to those for which the constraint vector lies either in a specific direction or within any direction in a specific plane. These two subclasses of constraint are labelled one- and two-dimensional in the classification proposed by Pipkin (1976).

The constraint of restricted shear is a mathematical idealization used to model a class of materials within which a preferred plane exists. Within this plane a large shear stress is required to produce a small shear strain. We now consider materials for which the modulus of shear within this preferred plane is an order of magnitude larger than any other elastic modulus of the material. The relaxation of the constraint enables us to employ a more physically-realistic model and also helps elucidate the singularities occurring on the slowness surface of the constrained body. A similar relaxation of a constraint was employed by Scott (1986) to examine the differences between the slowness surfaces of compressible and incompressible elastic bodies.

The discussion of nearly-constrained elastic materials will follow the pattern developed by Rogerson (1987). Consider an elastic body within which a preferred plane, defined by the vectors $\mathbf{a}$ and $\mathbf{b}$, exists within which the modulus of shear G is an order of magnitude larger than any other elastic modulus of the material. It will also be assumed that the angle between the two directions is very close to $\pi/2$, i.e. $(\mathbf{a}\cdot\mathbf{b})$ is assumed small. As the material approaches the limit of the constraint

$$G \to \infty, \quad (\mathbf{a}\cdot\mathbf{b}) \to 0 \quad (15)$$

in such a way that

$$G(\mathbf{a}\cdot\mathbf{b}) \rightarrow S, \tag{16}$$

where S is an arbitrary shear stress in the preferred plane brought into play by the restriction in shear. This shear stress is not determined by the deformation but must ultimately be chosen so that the equations of motion and all boundary conditions are satisfied. For a material with a high modulus of shear in the plane of $\mathbf{a}$ and $\mathbf{b}$ a possible strain energy function is of the form

$$W(\mathbf{F}) = W_0(\mathbf{F}) + \tfrac{1}{2}G(\mathbf{a}\cdot\mathbf{b})^2, \tag{17}$$

in which $W_0(\mathbf{F})$ is a strain energy function evaluated at $(\mathbf{a}\cdot\mathbf{b}) = 0$. The total Cauchy stress generated by $W(\mathbf{F})$ may upon invoking eqn (3) be shown to take the form

$$\sigma_{ij} = \sigma^0_{ij} + G(\mathbf{a}\cdot\mathbf{b})(a_i b_j + b_i a_j), \tag{18}$$

in which $\boldsymbol{\sigma}^0$ is the stress generated by $W_0(\mathbf{F})$. Equation (4) may now be employed to show that the elasticity tensor associated with the strain energy function (17) is given by

$$B_{ijkl} = B^0_{ijkl} + G(\mathbf{a}\cdot\mathbf{b})B^c_{ijkl} + G(a_i b_j + b_i a_j)(a_k b_l + a_l b_k), \tag{19}$$

where $\boldsymbol{B}^0$ is that part of $\boldsymbol{B}$ generated by $W_0(\mathbf{F})$ and $\boldsymbol{B}^c$ is the constraint contribution. It is now possible to use eqn $(8)_1$ to obtain the expression for the acoustical tensor

$$Q_{jl} = Q^1_{jl} + G[(\mathbf{a}\cdot\mathbf{n})b_j + (\mathbf{b}\cdot\mathbf{n})a_j][(\mathbf{a}\cdot\mathbf{n})b_l + (\mathbf{b}\cdot\mathbf{n})a_l], \tag{20}$$

where $\mathbf{Q}^1$ is the contribution to the acoustical tensor arising from the first two terms on the right-hand side of eqn (19). As the limit to the constraint specified by eqns (15) and (16) is approached, the Cauchy stress given in eqn (18) tends to that associated with the constrained problem shown implicitly in eqns (2) and (3). The arbitrary scalar multiplier α which was introduced in the pseudo-strain energy function (2) is now interpreted as an arbitrary shear stress in the preferred plane. It is also noted that as $G \rightarrow \infty$ the tensor $\mathbf{Q}^1$ introduced in eqn (20) tends to the constrained acoustical tensor. The second term on the right-hand side of eqn (20) is a part of the acoustical tensor which has no counterpart in the constrained case. This term is therefore of particular interest as the limit $G \rightarrow \infty$ is approached.

The classical unconstrained propagation condition may be formulated as the eigenvalue problem

$$Q_{jl}e_l - \lambda e_j = 0, \tag{21}$$

in which for convenience λ has been used in place of ρv^2, see Truesdell and Noll (1965, p. 71). If the acoustical tensor appropriate to our nearly-constrained material shown in eqn (20) is inserted into this propagation condition, we obtain

$$Q^1_{jl}e_l + G[(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{e}) + (\mathbf{b}\cdot\mathbf{n})(\mathbf{a}\cdot\mathbf{e})][(\mathbf{a}\cdot\mathbf{n})b_j + (\mathbf{b}\cdot\mathbf{n})a_j] - \lambda e_j = 0. \tag{22}$$

It is now assumed that λ and $\mathbf{e}$ may be expanded in powers of the dimensionless parameter $\eta = \bar{G}/G$, thus

$$\lambda = \eta^{-1}\lambda_{-1} + \lambda_0 + \eta\lambda_1 + O(\eta^2), \tag{23}$$

$$\mathbf{e} = \mathbf{e}_0 + \eta\mathbf{e}^1 + \eta^2\mathbf{e}^2 + O(\eta^3), \tag{24}$$

in which $\bar{G}$ is the maximum value of the modulus of shear in any plane containing $\mathbf{c}$. It was asserted earlier that it is assumed that G is an order of magnitude larger than any other elastic modulus of the material and so it is reasonable to assume that η is a small dimensionless parameter. If expressions of the form (23) and (24) are inserted into the propagation condition (22), the two leading-order equations obtained are

$$\lambda_{-1}e_j^0-\bar{G}[(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{e}^0)+(\mathbf{b}\cdot\mathbf{n})(\mathbf{a}\cdot\mathbf{e}^0)][(\mathbf{a}\cdot\mathbf{n})b_j+(\mathbf{b}\cdot\mathbf{n})a_j]=0, \tag{25}$$

$$Q_{jl}^1e_l^0+\bar{G}[(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{e}^1)+(\mathbf{b}\cdot\mathbf{n})(\mathbf{a}\cdot\mathbf{e}^1)][(\mathbf{a}\cdot\mathbf{n})b_j+(\mathbf{b}\cdot\mathbf{n})a_j]-\lambda_0e_j^0-\lambda_{-1}e_j^1=0. \tag{26}$$

If the scalar product of the leading order equation (25) is taken with $\mathbf{e}^0$, we deduce that

$$\lambda_{-1}=\bar{G}[(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{e}^0)+(\mathbf{b}\cdot\mathbf{n})(\mathbf{a}\cdot\mathbf{e}^0)]^2, \tag{27}$$

where it has been assumed that $\mathbf{e}^0\cdot\mathbf{e}^0=1$. This assumption, together with the assumption that $\mathbf{e}^0\cdot\mathbf{e}^1=0$, are made because of the fact that $\mathbf{e}$ is a unit vector. From eqn (27) it is concluded that there are two distinct cases, namely $\lambda_{-1}=0$ and $\lambda_{-1}\neq 0$, which will now be discussed in turn.

(a) $\lambda_{-1}=0$

Equation (14) may be used in conjunction with eqn (27) to show that $\lambda_{-1}=0$ when $\mathbf{e}^0$ is perpendicular to the constraint vector $\boldsymbol{\nu}$. In this case the second-order equation (26) reduces to

$$Q_{jl}^1e_l^0+\bar{G}(\boldsymbol{\nu}\cdot\mathbf{e}^1)\nu_j-\lambda_0e_j^0=0. \tag{28}$$

If this is compared with eqns (7) and (8), it is seen that $(\lambda_0^i, \mathbf{e}_i^0, i=1,2)$ are the same solutions as those of the associated constrained problem. Furthermore, these eigenvalues and associated eigenvectors are only $O(\eta)$ different from those of the constrained problem and tend to the constrained results as $\eta\to 0$, i.e. $G\to\infty$. An exceptional case arises in the case when $\mathbf{n}\to\mathbf{c}$, which will be discussed in the next section.

(b) $\lambda_{-1}\neq 0$

In the case $\lambda_{-1}\neq 0$ we are able to deduce from eqn (25) that

$$e_j^0=[(\mathbf{a}\cdot\mathbf{n})b_j+(\mathbf{b}\cdot\mathbf{n})a_j][(\mathbf{a}\cdot\mathbf{n})^2+(\mathbf{b}\cdot\mathbf{n})^2]^{-1/2}\equiv\hat{\nu}_j, \tag{29}$$

$$\lambda_{-1}=\bar{G}[(\mathbf{a}\cdot\mathbf{n})^2+(\mathbf{b}\cdot\mathbf{n})^2]\equiv\bar{G}(\boldsymbol{\nu}\cdot\boldsymbol{\nu}), \tag{30}$$

and from the second-order equation (26) that

$$Q_{jl}^1e_l^0+\bar{G}(\boldsymbol{\nu}\cdot\mathbf{e}^1)\nu_j-\lambda_0\hat{\nu}_j-\bar{G}(\boldsymbol{\nu}\cdot\boldsymbol{\nu})e_j^1=0. \tag{31}$$

In eqns (29)–(31) $\hat{\boldsymbol{\nu}}$ is a unit vector parallel to the constraint vector (14). It has already been stated that a consequence of $\mathbf{e}$ being a unit vector is that $\mathbf{e}^0\cdot\mathbf{e}^1=0$. For this particular case it is observed from eqn (29) that $\hat{\boldsymbol{\nu}}\cdot\mathbf{e}^1=0$. With this in mind the scalar product of eqn (31) is taken with $\hat{\boldsymbol{\nu}}$ to deduce that

$$\lambda_0=\hat{\nu}_jQ_{jl}^1\hat{\nu}_l. \tag{32}$$

We then conclude that the leading-order polarization vector is parallel to the constraint vector and then, making use of eqn (23), conclude that the corresponding eigenvalue takes the approximate form:

$$\lambda=G[(\mathbf{a}\cdot\mathbf{n})^2+(\mathbf{b}\cdot\mathbf{n})^2]+\hat{\boldsymbol{\nu}}\cdot(\mathbf{Q}^1\hat{\boldsymbol{\nu}})+O(\eta). \tag{33}$$

This equation enables us to deduce that λ is $O(G)$ provided $[(\mathbf{a}\cdot\mathbf{n})^2+(\mathbf{b}\cdot\mathbf{n})^2]$ is $O(1)$. It is possible for some $\mathbf{n}$ that the leading-order term of eqn (33) becomes $O(1)$ and is then comparable in magnitude with the second-order term of the expansion, explicitly this occurs when

$$(\mathbf{a}\cdot\mathbf{n})^2+(\mathbf{b}\cdot\mathbf{n})^2 = \mathrm{O}(G^{-1}). \quad (34)$$

This phenomenon will only occur when $\mathbf{n}$ is close to the direction of the normal to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

We now have all the necessary information required to determine the behaviour of the slowness surface as $G \to \infty$. It has already been established in Section 3(ii)(a) that when $\lambda_{-1} = 0$ the leading-order eigenvalues are exactly those pertaining to the corresponding constrained problem. Furthermore, as $G \to \infty$ the two associated outer sheets of the slowness surface will in general tend to the two corresponding sheets of the constrained surface. The case of waves propagating either in, or close to, the direction $\mathbf{n} = \mathbf{c}$ is of particular interest. From eqn (29) it is deduced that in the limit $\mathbf{n} = \mathbf{c}$ the unit constraint vector $\hat{\mathbf{v}}$ is undefined. For any plane containing $\mathbf{c}$, however, the limit of $\hat{\mathbf{v}}$ as $\mathbf{n} \to \mathbf{c}$ may be set equal to the constant value $\hat{\mathbf{v}}$ has taken for all $\mathbf{n}$ within that plane. The result is that the eigenvalue limit is not a true limit but rather a directionally-dependent one, which varies continuously as $\hat{\mathbf{v}}$ takes all possible values satisfying $\hat{\mathbf{v}}\cdot\mathbf{c} = 0$. This directionally-dependent type of limit has been observed in exceptional directions of incompressible, fibre-reinforced elastic solids, see Scott (1978).

For the second case $\lambda_{-1} \neq 0$ the eigenvalue tends to infinity as $G \to \infty$ in all directions except those in, or close to, $\mathbf{c}$. It has been shown by Rogerson (1987) that if a perturbation parameter of the form we have used is employed it is naturally replaced in the analysis by another parameter. This parameter ϵ is related to η in this case through the relation

$$\epsilon = \eta[(\mathbf{a}\cdot\mathbf{n})^2+(\mathbf{b}\cdot\mathbf{n})^2]. \quad (35)$$

It is then clear that the perturbation schemes are only valid provided $[(\mathbf{a}\cdot\mathbf{n})^2+(\mathbf{b}\cdot\mathbf{n})^2]$ is $\mathrm{O}(1)$, i.e. the direction of wave propagation is neither in, nor close to, $\mathbf{c}$. The slowness surface of an elastic body with high modulus of shear in a preferred plane then consists of three distinct sheets. The two outer sheets are very similar to the associated sheets of the constrained body and in general tend to them as $G \to \infty$. The innermost sheet is a thin pencil shape directed along the normal to the preferred plane.

4. ENERGY PROPAGATION

An examination of energy changes accompanying small-amplitude motions in elastic bodies subject to a general system of internal constraints and arbitrary homogeneous pre-strain has been carried out by Chadwick *et al.* (1985). Essentially using a pseudo-strain energy function of the form (2), a set of energy relations analogous to well-known results for elastic bodies free from constraints and pre-strain were derived. In order to do this both the energy flux vector and total energy density were partitioned into components termed interaction and incremental. These components both satisfy individual energy relations and admit distinct interpretations. The interaction components contain the equilibrium stress in B_r, while the incremental components persist even when the pre-strain is absent. It is these incremental components which are of primary relevance to the transport of energy in small-amplitude motions. We now consider the incremental components for materials within which a preferred plane with high modulus of shear exists.

The energy flux vector associated with small-amplitude motions is given by

$$J_i = -\sigma_{ij}\dot{U}_j. \quad (36)$$

The incremental energy flux vector is of the form (36) with σ interpreted as the Cauchy stress remaining if the stress in B_r were zero. Equation (36) may then be written in a form involving the fourth-order elasticity tensor to show that

$$J_i = -B_{ijkl} U_{l,k} \dot{U}_j, \tag{37}$$

in which the tensor $\boldsymbol{B}$ is given by eqn (19). Similarly, the incremental total energy density ε may be expressed in the form

$$\varepsilon = \tfrac{1}{2}\dot{U}_q \dot{U}_q + \tfrac{1}{2}\rho^{-1} B_{ijkl} U_{j,i} U_{l,k}. \tag{38}$$

ε being the sum of kinetic and internal energy. With a combination of the propagation condition (22) and the wave form (6) the kinetic energy K is given by

$$K = \tfrac{1}{2}\rho^{-1}[(\mathbf{Q}^1\mathbf{e})\cdot\mathbf{e} + G[(\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{e}) + (\mathbf{b}\cdot\mathbf{n})(\mathbf{a}\cdot\mathbf{e})]^2]\varphi'^2. \tag{39}$$

The second term on the right-hand side of eqn (38) is the internal energy density. Upon invoking eqns $(6)_1$ and (19) it is possible to deduce that the internal energy density and kinetic energy are equipartitioned.

A natural way to define an incremental energy velocity is through the relation

$$V_i = (\rho\varepsilon)^{-1} J_i. \tag{40}$$

Use may now be made of eqns (37), (38) and (19) to express $\mathbf{V}$ as

$$V_i = (\rho v)^{-1}[B^1_{ijkl} + G(a_i b_j + b_i a_j)(a_k b_l + b_k a_l)]e_j n_k, \tag{41}$$

in which $\boldsymbol{B}^1$ consists of the first two terms on the right-hand side of eqn (19). The propagation condition (22) is now used to verify that the energy velocity equals the ray velocity and that its projection on the wave normal equals the wave speed v. We have now verified the following results: (i) the kinetic energy and incremental energy density are equipartitioned, (ii) the incremental energy velocity is equal to the ray velocity, and (iii) the projection of the ray velocity on the wave normal is equal to the wave speed v. These three results are well-known results in the theory of unconstrained linear elastic wave propagation, see e.g. Schouten (1951, Chapter, Section 7).

We now consider the kinetic energy and energy velocity of the wave associated with the innermost sheet of the slowness surface. It is of particular interest to do this and let $G \to \infty$ because this wave has no counterpart in the constrained theory. Upon recalling that $\lambda = \rho v^2$ it is possible to use eqn (33) to obtain the expansion for v:

$$v = \{\rho^{-1} G[(\mathbf{a}\cdot\mathbf{n})^2 + (\mathbf{b}\cdot\mathbf{n})^2]\}^{1/2} + \mathrm{O}(G^{-1/2}). \tag{42}$$

In order that this expansion remains valid, attention will be restricted to the cases when $[(\mathbf{a}\cdot\mathbf{n})^2 + (\mathbf{b}\cdot\mathbf{n})^2]$ is O(1). The expansion for v and the fact that to within $\mathrm{O}(\eta)\mathbf{e} = \hat{\mathbf{v}}$ are now used to express the kinetic energy (39) in the form

$$K = \tfrac{1}{2}\rho^{-1}[G[(\mathbf{a}\cdot\mathbf{n})^2 + (\mathbf{b}\cdot\mathbf{n})^2] + (\mathbf{Q}^1\hat{\mathbf{v}})\cdot\hat{\mathbf{v}} + \mathrm{O}(G^{-1})]\varphi'^2, \tag{43}$$

where $\hat{\mathbf{v}}$ is given explicitly by eqn (29). The magnitude of the kinetic energy shown in eqn (43) is governed by the magnitude of $G\varphi'^2$. In order to investigate the behaviour of this term it is necessary to invert transform solutions. This interesting point, both for this and other internal constraints, will be fully explored at some future date.

We shall now turn our attention to the incremental energy velocity. The expansion (42) may be employed in conjunction with eqns (29) and (41) to obtain an expansion for $\mathbf{V}$. The leading-order term of this expansion $\mathbf{V}^0$ is given by

$$V^0_i = (\rho^{-1}G)^{1/2}[(\mathbf{a}\cdot\mathbf{n})a_i + (\mathbf{b}\cdot\mathbf{n})b_i][(\mathbf{a}\cdot\mathbf{n})^2 + (\mathbf{b}\cdot\mathbf{n})^2]^{-1/2}. \tag{44}$$

As $G \to \infty$ the magnitude of this vector also tends to infinity. Furthermore, the scalar product of eqn (44) may be taken with $\mathbf{n}$ to deduce that

$$\mathbf{V}^0 \cdot \mathbf{n} = \{\rho^{-1} G[(\mathbf{a}\cdot\mathbf{n})^2 + (\mathbf{b}\cdot\mathbf{n})^2]\}. \tag{45}$$

This, as expected, shows that the projection of $\mathbf{V}^0$ along the wave normal is equal to the leading-order term of the wave speed expansion (42). It is also of interest to note that $\mathbf{V}^0$ always lies within the preferred plane. Geometrically this result is expected because it is well known that $\mathbf{V}$ is normal to the slowness surface. In Section 3 it was shown that the appropriate sheet of the slowness surface is a thin pencil shape normal to the preferred plane. The normal to this sheet of the slowness surface is, to leading order, any direction within the preferred plane.

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